

# SELECTIVE ORDERS IN CENTRAL SIMPLE ALGEBRAS

BENJAMIN LINOWITZ AND THOMAS R. SHEMANSKE

**ABSTRACT.** Let  $B$  be a central simple algebra of degree  $n$  over a number field  $K$ , and let  $L/K$  be a field extension of degree  $n$  which embeds in  $B$ . The question of which of the isomorphism classes of maximal orders in  $B$  admit an embedding of an  $\mathcal{O}_K$ -suborder  $\Omega$  of  $\mathcal{O}_L$  is the question of selectivity of the order  $\Omega$ . In this paper we are concerned with algebras of degree  $n \geq 3$ , and continue the work of several authors to characterize the degree to which and conditions under which selectivity will occur. We take a local approach via the theory of affine buildings which provides explicit information about the structure of local embeddings, and leverage that information to produce a global characterization. We clarify that selectivity can never occur in an algebra which is a division algebra at a finite prime as well as construct representatives of the isomorphism classes of maximal orders which do admit embeddings of  $\Omega$ . An example of selectivity in an algebra with partial ramification is provided.

## 1. INTRODUCTION

Let  $K$  be a number field and  $B$  a central simple  $K$ -algebra. For each prime  $\nu$  of  $K$  the Wedderburn structure theorem implies that there is an isomorphism of local algebras  $B_\nu = B \otimes_K K_\nu \cong M_{\kappa_\nu}(D_\nu)$ , where  $D_\nu$  is a central simple division algebra over  $K_\nu$  with index  $m_\nu$ . The Albert-Brauer-Hasse-Noether theorem allows one to easily determine when a field extension of  $K$  embeds into  $B$ :

**Theorem.** (*ABNH*) *Let  $K$  be a number field and  $B$  a central simple algebra of dimension  $n^2$  over its center  $K$ . Suppose that  $[L : K] = n$ . There is an embedding of  $L/K$  into  $B$  if and only if for each prime  $\nu$  of  $K$  and for all primes  $\mathfrak{P}$  of  $L$  lying above  $\nu$ ,  $m_\nu \mid [L_\mathfrak{P} : K_\nu]$ .*

It is natural to seek an integral analogue of the Albert-Brauer-Hasse-Noether theorem. Let  $K, L$  be as above with rings of integers  $\mathcal{O}_K$  and  $\mathcal{O}_L$  respectively. Suppose that  $K \subset L \subset B$ . For any  $\mathcal{O}_K$ -order  $\Omega \subseteq \mathcal{O}_L$ , there exists an  $\mathcal{O}_K$ -order  $\mathcal{R}$  of rank  $n$  in  $B$  which contains  $\Omega$  (see page 131 of [19]), and it is clear that any order in the same isomorphism class of  $\mathcal{R}$  admits an embedding of  $\Omega$ . An integral analogue of the Albert-Brauer-Hasse-Noether theorem should address the question of whether the orders in  $B$  which are not isomorphic to  $\mathcal{R}$  but lie in the same genus (locally isomorphic at all finite primes) also admit embeddings of  $\Omega$ .

---

*Date:* March 2, 2013.

*2010 Mathematics Subject Classification.* Primary 11R54; Secondary 11S45, 20E42.

*Key words and phrases.* Order, central simple algebra, affine building, embedding.

Since it is clear that either every order in an isomorphism class of orders in  $B$  admits an embedding of  $\Omega$  or none at all do, it is a convenient abuse of language to say that an isomorphism class either admits or does not admit an embedding of  $\Omega$ . In the case that not all isomorphism classes of maximal orders admit an embedding of  $\Omega$ , we follow Chinburg and Friedman [10] and call  $\Omega$  selective.

The first integral refinement of the Albert-Brauer-Hasse-Noether theorem was due to Chevalley [9]. Let  $K$  be a number field,  $B = M_n(K)$  and  $L/K$  be a degree  $n$  extension of fields. Chevalley's elegant result is:

**Theorem.** (*Chevalley*) *The ratio of the number of isomorphism classes of maximal orders in  $B = M_n(K)$  into which  $\mathcal{O}_L$  can be embedded to the total number of isomorphism classes of maximal orders is  $[\hat{K} \cap L : K]^{-1}$  where  $\hat{K}$  is the Hilbert class field of  $K$ .*

In 1999, Chinburg and Friedman [10] considered quaternion algebras  $B$  satisfying the Eichler condition and proved that the proportion of isomorphism classes of maximal orders admitting an embedding of a commutative, quadratic  $\mathcal{O}_K$ -order  $\Omega$  contained in an embedded quadratic extension of the center is either  $1/2$  or  $1$ . Their result was later extended to Eichler orders of arbitrary level by Chan and Xu [8] and independently by Guo and Qin [12]. Maclachlan [17] considered Eichler orders of square-free level and proved the analogous theorem for optimal embeddings. The first author of this paper gave a number of criteria [15] which imply that  $\Omega$  is not selective with respect to the genus of any fixed order  $\mathcal{R}$  of  $B$ .

Before continuing with the progress on this problem, we note that selectivity has many interesting applications to hyperbolic geometry. Suppose that  $K \neq \mathbb{Q}$  is a totally real number field,  $B$  is a quaternion algebra over  $K$  in which a unique real prime splits, and that  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are maximal orders of  $B$  representing distinct isomorphism classes. Vigneras [22] used  $\mathcal{R}_1$  and  $\mathcal{R}_2$  to construct non-isometric compact hyperbolic 2-manifolds  $M_1$  and  $M_2$ . She further showed that a sufficient condition for  $M_1$  and  $M_2$  to be isospectral with respect to the Laplace-Beltrami operator was the nonexistence of a selective order  $\Omega$  which embedded into exactly one of  $\{\mathcal{R}_1, \mathcal{R}_2\}$ . Vigneras' original examples of isospectral but not isometric hyperbolic 2-manifolds have enormous genus: 100,801. Recently Doyle, Voight and the first author of this paper [11] have applied the results of [10],[8],[12] in order to produce isospectral but not isometric hyperbolic 2-orbifolds with underlying surface of genus 0, and have proven that these orbifolds are minimal among all isospectral but not isometric surfaces arising from maximal arithmetic Fuchsian groups. Although we do not consider applications to hyperbolic geometry in this paper, it is likely our results can be applied in order to prove the isospectrality of lattices in more general symmetric spaces.

The first work beyond Chevalley's in the non-quaternion setting was by Arenas-Carmona [1]. The setting was a central simple algebra  $B$  over a number field  $K$  of dimension  $n^2$ ,  $n \geq 3$  with the proviso that at each finite prime  $\nu$  of  $K$ ,  $B_\nu$  is either  $M_n(K_\nu)$  or a division algebra, that is  $\nu$  either splits in  $B$  or is totally ramified. He considered embeddings of the ring of integers  $\mathcal{O}_L$  of an extension  $L/K$  of degree  $n$  into maximal orders of  $B$  and establishes a

proportion analogous to Chevalley's: his Theorem 1 says that the ratio of the number of isomorphism classes of maximal orders in  $B$  into which  $\mathcal{O}_L$  can be embedded to the total number of isomorphism classes of maximal orders is  $[\Sigma \cap L : K]^{-1}$  where  $\Sigma$  is a spinor class field. In Corollary 5.2, we show that if  $B_\nu$  is a division algebra for any prime  $\nu$ , this proportion is always one, meaning (that restricted to algebras with no partial ramification), we must have  $B_\nu = M_n(K_\nu)$  for every finite prime  $\nu$  in order to realize selective orders.

In [16], we considered central simple algebras  $B$  of odd prime degree  $p$ , but considered embedding nonmaximal orders  $\Omega \subset \mathcal{O}_L$  into maximal orders in  $B$  and proved analogous results (Theorem 3.3 of [16]) refined to indicate how the conductor of  $\Omega$  affects selectivity. In that paper we noted the first instance of Corollary 5.2, precluding selectivity when  $B$  was a degree  $p$  division algebra.

Continuing with the history, in a recent preprint [2], Carmona extends his results on spinor class fields to:

**Theorem.** *Let  $\mathfrak{A}$  be a central simple algebra over the number field  $K$ . Let  $\mathfrak{H}$  be an arbitrary commutative order, Let  $\mathfrak{D}$  be a maximal order containing  $\mathfrak{H}$ . For every maximal ideal  $\wp$  in the ring of integers  $\mathcal{O}_K$  we let  $I_\wp$  be the only maximal two-sided ideal of  $\mathfrak{D}$  containing  $\wp 1_\mathfrak{D}$ , and let  $\mathbb{H}_\wp$  be the image of  $\mathfrak{H}$  in  $\mathfrak{D}/I_\wp$ . Let  $\mathbb{E}_\wp$  be the center of the ring  $\mathfrak{D}/I_\wp$ . Let  $t_\wp$  be the greatest common divisor of the dimensions of the irreducible  $\mathbb{E}_\wp$ -representations of the algebra  $\mathbb{E}_\wp \mathbb{H}_\wp$ . Then the representation field  $F(\mathfrak{H})$  is the maximal subfield  $F$ , of the spinor class field  $\Sigma$ , such that the inertia degree  $f_\wp(F/K)$  divides  $t_\wp$  for every place  $\wp$ .*

In this paper we have a number of goals. While we work in a setting slightly less general than [2], we produce results which nonetheless refine several of those in [2], give an explicit blueprint of how to realize the embeddings locally, and have applications both to and which extend beyond the specific question of selective orders. We first note that Corollary 5.2 and Example 5.3 provide significant clarification to the circumstances in which selectivity can occur, in particular refining the final corollaries of [2]. The key issue is one of ramification in the algebra.

A central simple algebra  $B$  of prime degree  $p \geq 2$  over a number field  $K$  can only split or be a division algebra at any finite prime of  $K$ , that is to say there is no partial ramification at the finite primes. When considering general central simple algebras of degree  $n \geq 3$  over  $K$ , partial ramification can occur. The results of [1] explicitly assume no partial ramification as do the final corollaries of [2] in stating results about selectivity. However, Corollary 5.2 says that selectivity never occurs if there is a finite prime  $\nu$  of  $K$  for which  $B_\nu$  is a division algebra, which is to say that an assumption of no partial ramification at the finite primes (in the context of selectivity) is exactly the statement that the algebra must split at all finite primes. Example 5.3 confirms selectivity is still possible in the case of partial ramification.

We begin the paper with a detailed investigation of a local embedding problem. Since a central simple algebra  $B/K$  is split at almost all primes of  $K$ , we consider the generic

case of embedding the global ring of integers  $\mathcal{O}_L$  into maximal orders in  $M_n(k)$  where  $k$  is a non-archimedean local field. Corollary 2.5 shows that this is really a local question:

**Corollary.** *Let  $B$  be a central simple algebra over a number field  $K$  of dimension  $n^2$ , and  $L$  a degree  $n$  field extension of  $K$  which embeds into  $B$ . Let  $\mathfrak{p}$  be a prime of  $K$  which splits in  $B$  and is unramified in  $L$ , and let  $\mathcal{E}_{\mathfrak{p}}$  a maximal order in  $B_{\mathfrak{p}}$ . Then the following are equivalent:*

- (1)  $\mathcal{O}_L \subset \mathcal{E}_{\mathfrak{p}}$
- (2)  $\mathcal{O}_{L_{\mathfrak{p}}} \subset \mathcal{E}_{\mathfrak{p}}$  for some prime  $\mathfrak{P}$  of  $L$  dividing  $\mathfrak{p}\mathcal{O}_L$ .
- (3)  $\bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{L_{\mathfrak{P}}} \subset \mathcal{E}_{\mathfrak{p}}$  where the sum is over all primes  $\mathfrak{P} \mid \mathfrak{p}\mathcal{O}_L$ .

Since every maximal order in  $M_n(k)$  can be identified with a vertex in the affine building for  $SL_n(k)$ , we are able to leverage both algebraic and geometric properties of buildings to give a very explicit characterization of these embeddings (Theorem 2.1):

**Theorem.** *Let  $B$  be a central simple algebra over a number field  $K$  of dimension  $n^2$  and  $L$  a degree  $n$  field extension of  $K$  which embeds into  $B$ . Let  $\mathfrak{p}$  be a prime of  $K$  which splits in  $B$  and is unramified in  $L$ , and write  $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1\mathfrak{P}_2 \cdots \mathfrak{P}_g$ , where  $f_i = f(\mathfrak{P}_i|\mathfrak{p})$  are the associated inertia degrees. Then  $\mathcal{O}_L$  is contained in the maximal orders of  $B_{\mathfrak{p}} \cong M_n(K_{\mathfrak{p}})$  represented by the homothety class  $[\mathcal{L}] = [a_1, \dots, a_n] \in \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1)$  if and only if there are  $\ell_i \in \mathbb{Z}$  such that  $[\mathcal{L}] = [\underbrace{\ell_1, \dots, \ell_1}_{f_1}, \underbrace{\ell_2, \dots, \ell_2}_{f_2}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{f_g}]$ .*

To apply these local results to the problem of selectivity, we next construct a set of representatives of the isomorphism classes of maximal orders in  $B$ , i.e., representatives of the genus of  $\mathcal{R}$ , using a local-global correspondence. We associate to  $\mathcal{R}$  a class field,  $K(\mathcal{R})$ , which for maximal orders  $\mathcal{R}$  is contained in the Hilbert class field of  $K$ . The class field corresponds to a class group derived from the normalizer of  $\mathcal{R}$ . We show that key to determining whether  $\mathcal{O}_L$  embeds into a maximal order  $\mathcal{E}$  in  $B$  is the arithmetic of  $L_0 = L \cap K(\mathcal{R})$ . More precisely, using the local characterization of maximal orders via affine buildings, we define an idelic analog,  $\delta(\mathcal{R}, \mathcal{E})$ , of the distance ideal (see [10]) between the fixed maximal order  $\mathcal{R}$  and any other maximal order  $\mathcal{E}$ . We prove (Theorem 4.1) that

**Theorem.**  *$\mathcal{E}$  admits an embedding of  $\mathcal{O}_L$  if and only if the idelic Artin symbol  $(\delta(\mathcal{R}, \mathcal{E}), L_0/K)$  is trivial in  $\text{Gal}(L_0/K)$ .*

From this we deduce a selectivity result, Corollary 4.2:

**Corollary.** *The ratio of the number of isomorphism classes of maximal orders in  $B$  which admit an embedding of  $\mathcal{O}_L$  to the total number of isomorphism classes of maximal orders is  $[L_0 : K]^{-1}$  where  $L_0 = K(\mathcal{R}) \cap L$ .*

This corollary has the same shape as many previous results giving the proportion (in terms of an index of fields) of the number of isomorphism classes of maximal orders in  $B$  which admit an embedding of  $\mathcal{O}_L$ , but differs in the choice of class field which is leveraged

to produce the proportion: not the Hilbert class field of Chevalley, nor the spinor class field of Arenas-Carmona. On the other hand, it is true that for maximal orders  $\mathcal{R}$ , the class field  $K(\mathcal{R})$  is a subfield of the Hilbert class field of  $K$  having exponent  $n$ . The above corollary implies Chevalley's theorem (in the case that  $B = M_n(K)$ ) and should be viewed as a generalization of Chevalley's result to arbitrary central simple algebras of finite degree. But the concreteness of our construction allows us to sharpen the result further, by computing the index in specific cases: Corollary 5.2 shows that if  $B_\nu$  is a division algebra at any finite prime, then there is no selectivity (i.e., the index is one). In particular if  $\Omega \subseteq \mathcal{O}_L$  is any  $\mathcal{O}_K$ -order, then  $\Omega$  is never selective; that is, every maximal order in  $B$  admits an embedding of  $\Omega$ . We conclude with Example 5.3 produced using Magma [6], which demonstrates that selectivity in an algebra with partial ramification is still possible.

## 2. LOCAL EMBEDDINGS VIA BUILDINGS

Let  $L/K$  be a degree  $n$  extension of number fields,  $B/K$  a central simple algebra of dimension  $n^2$  which contains  $L$ , and  $\mathfrak{p}$  a prime of  $K$  which splits in  $B$  and is unramified in  $L$ . In this section we characterize explicitly all the maximal orders of  $B_\mathfrak{p}$  which admit an embedding of  $\mathcal{O}_L$ .

Since  $\mathcal{O}_L$  has rank  $n$  over  $K$ , any embedding  $\varphi$  of  $\mathcal{O}_L$  into a maximal order  $\mathcal{E} \subset B_\mathfrak{p}$  extends to an inner automorphism of  $B_\mathfrak{p}$  by Skolem-Noether, thus  $\mathcal{E}$  contains a conjugate of  $\mathcal{O}_L$ , or equivalently,  $\mathcal{O}_L$  is contained in a conjugate of  $\mathcal{E}$ . We therefore begin by characterizing those maximal orders which contain  $\mathcal{O}_L$ .

Fix a maximal order  $\mathcal{R}$  of  $B$  containing  $\mathcal{O}_L$ . The prime  $\mathfrak{p}$  splits in  $B$ , so that  $B_\mathfrak{p} \cong M_n(K_\mathfrak{p})$  hence all maximal orders are conjugate ((17.3), (17.4) of [19]). By a choice of basis we may assume that  $B_\mathfrak{p}$  is identified with  $M_n(K_\mathfrak{p})$  so that  $\mathcal{R}_\mathfrak{p} = M_n(\mathcal{O}_\mathfrak{p})$ .

Write  $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1\mathfrak{P}_2 \cdots \mathfrak{P}_g$  as a product of primes in  $L$ , and let  $f_i = f(\mathfrak{P}_i|\mathfrak{p})$  be the associated inertia degrees. Of course we have  $f_1 + \cdots + f_g = n$ , and

$$(1) \quad L \otimes_K K_\mathfrak{p} \cong \bigoplus_{i=1}^g L_{\mathfrak{P}_i} \hookrightarrow \bigoplus_{i=1}^g M_{f_i}(K_\mathfrak{p}) \hookrightarrow M_n(K_\mathfrak{p}).$$

In particular, since all the maximal orders of  $M_{f_i}(K_\mathfrak{p})$  are conjugate, by a change of basis we adjust the embeddings  $L_{\mathfrak{P}_i} \hookrightarrow M_{f_i}(K_\mathfrak{p})$  so that  $\mathcal{O}_{\mathfrak{P}_i} \hookrightarrow M_{f_i}(\mathcal{O}_\mathfrak{p})$ . Thus

$$(2) \quad \mathcal{O}_L \subset \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_\mathfrak{p} \hookrightarrow \bigoplus_{i=1}^g \mathcal{O}_{\mathfrak{P}_i} \hookrightarrow \bigoplus_{i=1}^g M_{f_i}(\mathcal{O}_\mathfrak{p}) \hookrightarrow M_n(\mathcal{O}_\mathfrak{p}).$$

Fix a uniformizing parameter  $\pi$  of  $\mathcal{O}_\mathfrak{p}$ , and let  $D_k^\ell = \text{diag}(\underbrace{\pi^\ell, \dots, \pi^\ell}_k, 1, \dots, 1) \in M_n(K_\mathfrak{p})$ .

Thus

$$(3) \quad \mathcal{R}(k, \ell) := D_k^\ell M_n(\mathcal{O}_\mathfrak{p}) D_k^{-\ell} = \begin{pmatrix} M_k(\mathcal{O}_\mathfrak{p}) & \mathfrak{p}^\ell M_{k \times n-k}(\mathcal{O}_\mathfrak{p}) \\ \mathfrak{p}^{-\ell} M_{n-k \times k}(\mathcal{O}_\mathfrak{p}) & M_{n-k}(\mathcal{O}_\mathfrak{p}) \end{pmatrix} \subset M_n(K_\mathfrak{p}).$$

Note that  $\mathcal{R}(0, \ell) = \mathcal{R}(n, \ell) = \mathcal{R}(k, 0) = \mathcal{R}_{\mathfrak{p}} = M_n(\mathcal{O}_{\mathfrak{p}})$ . From equations (2),(3) above, it is evident that for all  $\ell_1, \dots, \ell_g \in \mathbb{Z}$ ,

$$(4) \quad \mathcal{O}_L \subset \mathcal{R}(f_1, \ell_1) \cap \mathcal{R}(f_1 + f_2, \ell_2) \cap \dots \cap \mathcal{R}(f_1 + \dots + f_g, \ell_g),$$

that is,

$$(5) \quad \mathcal{O}_L \subset \bigcap_{\ell_i \in \mathbb{Z}} [\mathcal{R}(f_1, \ell_1) \cap \mathcal{R}(f_1 + f_2, \ell_2) \cap \dots \cap \mathcal{R}(f_1 + \dots + f_g, \ell_g)] = \bigoplus_{i=1}^g M_{f_i}(\mathcal{O}_{\mathfrak{p}}).$$

We now translate this to the language of affine buildings. We have picked a basis  $\{\omega_1, \dots, \omega_n\}$  of  $K_{\mathfrak{p}}^n$  (and hence in particular fixed an apartment of the building associated to  $SL_n(K_{\mathfrak{p}})$ ), so that with respect to this basis,  $\mathcal{R}_{\mathfrak{p}} = M_n(\mathcal{O}_{\mathfrak{p}}) = \text{End}(\Lambda)$ , where  $\Lambda = \bigoplus_{i=1}^n \mathcal{O}_{\mathfrak{p}} \omega_i$ . Also we have  $\mathcal{R}(k, \ell) = \text{End}(\mathcal{M}(k, \ell))$  where  $\mathcal{M}(k, \ell) = \bigoplus_{i=1}^k \mathcal{O}_{\mathfrak{p}} \pi^{\ell} \omega_i \oplus \bigoplus_{i=k+1}^n \mathcal{O}_{\mathfrak{p}} \omega_i$  and  $\pi$  is our fixed uniformizer in  $\mathcal{O}_{\mathfrak{p}}$ . As usual, this maximal order in  $B_{\mathfrak{p}}$  can be represented by the homothety class of the lattice  $\mathcal{M}(k, \ell)$ ,  $[\mathcal{M}(k, \ell)] := [\underbrace{\ell, \dots, \ell}_k, 0, \dots, 0] \in \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$ .

Observe that  $\mathcal{R}(k, \ell)$  has type  $k\ell \pmod{n}$  (see [7]).

With the notation fixed as above, we characterize precisely which maximal orders in this apartment contain  $\mathcal{O}_L$ .

**Theorem 2.1.** *Let  $B$  be a central simple algebra over a number field  $K$  of dimension  $n^2$  and  $L$  a degree  $n$  field extension of  $K$  which embeds into  $B$ . Let  $\mathfrak{p}$  be a prime of  $K$  which splits in  $B$  and is unramified in  $L$ , and write  $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1 \mathfrak{P}_2 \dots \mathfrak{P}_g$ , where  $f_i = f(\mathfrak{P}_i | \mathfrak{p})$  are the associated inertia degrees. Then  $\mathcal{O}_L$  is contained in the maximal orders of  $B_{\mathfrak{p}}$  represented by the homothety class  $[\mathcal{L}] = [a_1, \dots, a_n] \in \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$  if and only if there are  $\ell_i \in \mathbb{Z}$  such that  $[\mathcal{L}] = [\underbrace{\ell_1, \dots, \ell_1}_{f_1}, \underbrace{\ell_2, \dots, \ell_2}_{f_2}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{f_g}]$ .*

The following corollary is immediate.

**Corollary 2.2.**  *$\mathcal{O}_L$  is contained in maximal orders in  $B_{\mathfrak{p}}$  of types  $td$  where  $d = \gcd(f_1, \dots, f_g) \mid n$  and  $t = 1, 2, \dots, n/d$ . In particular in a standard fundamental chamber containing  $\mathcal{R}_{\mathfrak{p}} \leftrightarrow [0, \dots, 0]$ ,  $\mathcal{O}_L$  is contained in the maximal orders corresponding to the vertices  $v_0 = [0, \dots, 0]$ ,  $v_1 = [\underbrace{1, \dots, 1}_{f_1}, 0, \dots, 0]$ ,  $v_2 = [\underbrace{1, \dots, 1}_{f_1+f_2}, 0, \dots, 0]$ ,  $\dots$ ,*

$$v_{g-1} = [\underbrace{1, \dots, 1}_{f_1+\dots+f_{g-1}}, 0, \dots, 0].$$

*Remark 2.3.* Theorem 2.1 answers the question of which maximal orders contain  $\mathcal{O}_L$ . The answer to which orders admit an embedding of  $\mathcal{O}_L$  follows quite easily: If an order  $\mathcal{E}$  admits an embedding of  $\mathcal{O}_L$ , then a conjugate of  $\mathcal{E}$  contains  $\mathcal{O}_L$ . But conjugation just amounts to a change of basis and hence a change of apartment in the building, so we can simply translate the results for containment to those of embedding by conjugation.

To proceed, we need a technical lemma.

**Lemma 2.4.** *Let  $R$  be the valuation ring of  $\mathfrak{p}$  in  $K$ , and  $S$  its integral closure in  $L$ . Suppose that  $\mathcal{E}_{\mathfrak{p}}$  is a maximal order in  $B_{\mathfrak{p}} = M_n(K_{\mathfrak{p}})$  and that  $\mathcal{O}_L \subset \mathcal{E}_{\mathfrak{p}}$ . Then  $S \subset \mathcal{E}_{\mathfrak{p}}$ .*

*Proof.* By Corollary 5.22 of [4],  $S$  is the intersection of all valuation rings of  $L$  which contain  $R$ . The valuation ring  $R$  is equal to the localization of  $\mathcal{O}_K$  at the prime ideal  $\mathfrak{p}$ , and the valuation rings of  $L$  which contain  $R$  are precisely the localizations of  $\mathcal{O}_L$  at the primes  $\mathfrak{P}_1, \dots, \mathfrak{P}_g$  of  $L$  which lie above  $\mathfrak{p}$ . By p43 of [19], the intersection of these localizations,  $S$ , is equal to the localization  $T^{-1}\mathcal{O}_L$ , where  $T = \mathcal{O}_L \setminus (\mathfrak{P}_1 \cup \dots \cup \mathfrak{P}_g)$ . To show  $S \subset \mathcal{E}_{\mathfrak{p}}$  we choose  $\alpha/\beta \in S$  with  $\alpha, \beta \in \mathcal{O}_L$  and  $\beta \notin \mathfrak{P}_1 \cup \dots \cup \mathfrak{P}_g$ . We are assuming  $\mathcal{O}_L \subset \mathcal{E}_{\mathfrak{p}} = \gamma_{\mathfrak{p}} M_n(\mathcal{O}_{\mathfrak{p}}) \gamma_{\mathfrak{p}}^{-1}$  for some  $\gamma_{\mathfrak{p}} \in B_{\mathfrak{p}}^{\times}$ , so write  $\alpha = \gamma_{\mathfrak{p}} A \gamma_{\mathfrak{p}}^{-1}$  and  $\beta = \gamma_{\mathfrak{p}} B \gamma_{\mathfrak{p}}^{-1}$  for  $A, B \in M_n(\mathcal{O}_{\mathfrak{p}})$ . Finally recall that  $\mathcal{O}_L \subset \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{p}} \hookrightarrow \bigoplus_{i=1}^g \mathcal{O}_{\mathfrak{P}_i} \hookrightarrow \bigoplus_{i=1}^g M_{f_i}(\mathcal{O}_{\mathfrak{p}}) \hookrightarrow M_n(\mathcal{O}_{\mathfrak{p}})$ , so that  $\det(\beta) = \prod_{i=1}^g N_{L_{\mathfrak{P}_i}/K_{\mathfrak{p}}}(\beta) = N_{L/K}(\beta)$ . As  $\beta \in \mathcal{O}_{\mathfrak{P}_i}^{\times}$  for each  $i$ , its local norm is in  $\mathcal{O}_{\mathfrak{p}}^{\times}$ . Thus  $\beta$  is a unit in  $\mathcal{E}_{\mathfrak{p}}$  from which the lemma follows.  $\square$

*Proof of Theorem.* Consider equation (5). We know that  $\mathcal{O}_L$  is contained in  $\mathcal{R}(f_1, \ell_1) \cap \mathcal{R}(f_1 + f_2, \ell_2) \cap \dots \cap \mathcal{R}(f_1 + \dots + f_g, \ell_g)$  for any choice of  $\ell_i \in \mathbb{Z}$ . These orders correspond to homothety classes of lattices  $[\mathcal{M}(f_1 + \dots + f_i, \ell_i)] = \ell_i [\mathcal{M}(f_1 + \dots + f_i, 1)] = \ell_i \underbrace{[1, \dots, 1, 0, \dots, 0]}_{f_1 + \dots + f_i}$  as an

element of  $\mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$ . In [5], it is shown that walks in an apartment are consistent with the natural group action on  $\mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$ , and since by [21] the intersection of any finite number of maximal orders (containing  $\mathcal{O}_{\mathfrak{p}}^n$ ) in an apartment is the same as the intersection of all the maximal orders in the convex hull they determine, we deduce that  $\mathcal{O}_L$  is contained in maximal orders corresponding to

$$[\mathcal{M}(f_1, \ell_1) + \mathcal{M}(f_1 + f_2, \ell_2) + \dots + \mathcal{M}(f_1 + \dots + f_g, \ell_g)] = \underbrace{[\ell_1 + \dots + \ell_g, \dots, \ell_1 + \dots + \ell_g]}_{f_1} \underbrace{[\ell_2 + \dots + \ell_g, \dots, \ell_2 + \dots + \ell_g]}_{f_2} \dots \underbrace{[\ell_g, \dots, \ell_g]}_{f_g}.$$

Since the  $\ell_i \in \mathbb{Z}$  are arbitrary, a simple change of variable ( $\ell_k + \dots + \ell_g \mapsto \ell_k$ ) shows that  $\mathcal{O}_L$  is contained in the maximal orders specified in the proposition. We now show these are the only maximal orders in the apartment which contain  $\mathcal{O}_L$ .

Suppose that  $\mathcal{O}_L$  is contained in a maximal order  $\Lambda(a_1, \dots, a_n)$  where

$$\Lambda(a_1, \dots, a_n) = \text{diag}(\pi^{a_1}, \dots, \pi^{a_n}) M_n(\mathcal{O}_{\mathfrak{p}}) \text{diag}(\pi^{a_1}, \dots, \pi^{a_n})^{-1} = \begin{pmatrix} \mathcal{O}_{\mathfrak{p}} & \mathfrak{p}^{a_1-a_2} & \mathfrak{p}^{a_1-a_3} & \dots & \mathfrak{p}^{a_1-a_n} \\ \mathfrak{p}^{a_2-a_1} & \mathcal{O}_{\mathfrak{p}} & \mathfrak{p}^{a_2-a_3} & \dots & \mathfrak{p}^{a_2-a_n} \\ \mathfrak{p}^{a_3-a_1} & \mathfrak{p}^{a_3-a_2} & \ddots & \dots & \mathfrak{p}^{a_3-a_n} \\ \vdots & \vdots & & \mathcal{O}_{\mathfrak{p}} & \vdots \\ \mathfrak{p}^{a_n-a_1} & \dots & & \mathfrak{p}^{a_n-a_{n-1}} & \mathcal{O}_{\mathfrak{p}} \end{pmatrix},$$

that is  $\Lambda(a_1, \dots, a_n)$  corresponds to the homothety class of the lattice  $[a_1, \dots, a_n]$  relative to our fixed basis  $\{\omega_1, \dots, \omega_n\}$  of  $K_{\mathfrak{p}}^n$ . By equation (5), we can reorder subsets of the basis  $\{\omega_1, \dots, \omega_{f_1}\}, \{\omega_{f_1+1}, \dots, \omega_{f_1+f_2}\}, \dots, \{\omega_{f_1+\dots+f_{g-1}+1}, \dots, \omega_n\}$  so that equation (5) remains valid and  $a_1 \leq \dots \leq a_{f_1}, a_{f_1+1} \leq \dots \leq a_{f_1+f_2}, \dots, a_{f_1+\dots+f_{g-1}+1} \leq \dots \leq a_n$ .

Now we assume that  $[a_1, \dots, a_n]$  is not of the form  $[\underbrace{\ell_1, \dots, \ell_1}_{f_1}, \underbrace{\ell_2, \dots, \ell_2}_{f_2}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{f_g}]$  for  $\ell_i \in \mathbb{Z}$ . Since we can permute the order in which we list the primes  $\mathfrak{P}_i$  of  $L$  lying above  $\mathfrak{p}$ , we may assume that there is an  $r_0$  with  $1 < r_0 \leq f_1$  so that  $a_1 = \dots = a_{r_0-1} < a_{r_0} \leq \dots \leq a_{f_1}$ . We already know that  $\mathcal{O}_L \hookrightarrow \bigoplus_{i=1}^g M_{f_i}(\mathcal{O}_{\mathfrak{p}}) \hookrightarrow M_n(\mathcal{O}_{\mathfrak{p}})$ , so we focus on the upper  $f_1 \times f_1$  block of  $\bigoplus_{i=1}^g M_{f_i}(\mathcal{O}_{\mathfrak{p}}) \cap \Lambda(a_1, \dots, a_n)$ . That intersection is contained in  $\Gamma_{\mathfrak{p}} := \begin{pmatrix} M_{r_0-1}(\mathcal{O}_{\mathfrak{p}}) & \pi^{a_1-a_{f_1}} M_{r_0-1 \times f_1-r_0+1}(\mathcal{O}_{\mathfrak{p}}) \\ \pi M_{f_1-r_0+1 \times r_0-1}(\mathcal{O}_{\mathfrak{p}}) & M_{f_1-r_0+1}(\mathcal{O}_{\mathfrak{p}}) \end{pmatrix} \cap M_{f_1}(\mathcal{O}_{\mathfrak{p}})$ .

Since  $\mathcal{O}_L \subset \Lambda(a_1, \dots, a_n)$ , Lemma 2.4 gives us that  $S$ , the integral closure of  $R = (\mathcal{O}_{\mathfrak{p}} \cap K)$  in  $L$ , is contained in  $\Lambda(a_1, \dots, a_n)$ . Thus  $S \otimes_R \mathcal{O}_{\mathfrak{p}} \subset \Lambda(a_1, \dots, a_n)$ . By Proposition II.4 of [20],  $S \otimes_R \mathcal{O}_{\mathfrak{p}} \cong \bigoplus_{i=1}^g \mathcal{O}_{\mathfrak{P}_i}$ , so  $\bigoplus_{i=1}^g \mathcal{O}_{\mathfrak{P}_i} \subset \Lambda(a_1, \dots, a_n)$ .

As  $\bigoplus_{i=1}^g \mathcal{O}_{\mathfrak{P}_i} \subset \bigoplus_{i=1}^g M_{f_i}(\mathcal{O}_{\mathfrak{p}})$ , we may assume that  $\mathcal{O}_{\mathfrak{P}_1} \subset \Gamma_{\mathfrak{p}}$ , from which we shall derive a contradiction. Since  $L_{\mathfrak{P}_1}/K_{\mathfrak{p}}$  is unramified, we can write  $\mathcal{O}_{\mathfrak{P}_1} = \mathcal{O}_{\mathfrak{p}}[\alpha]$  for some  $\alpha$  whose reduction modulo  $\mathfrak{p}$  generates the residue field extension (see Theorem 5.8 of [19]). In particular if  $h$  is the minimal polynomial of  $\alpha$  over  $K_{\mathfrak{p}}$ , then  $h \in \mathcal{O}_{\mathfrak{p}}[x]$  is irreducible of degree  $f_1$  and the reduction of  $h$  modulo  $\mathfrak{p}$  is irreducible. Now viewing  $\alpha$  as an element of  $\Gamma_{\mathfrak{p}}$ , we consider its characteristic polynomial  $\chi_{\alpha} \in \mathcal{O}_{\mathfrak{p}}[x]$ , also of degree  $f_1$ . Since  $h$  is irreducible, we have  $h \mid \chi_{\alpha}$  and so by degree arguments,  $h = \chi_{\alpha}$ . On the other hand  $\Gamma_{\mathfrak{p}}$ , viewed modulo  $\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ , is upper triangular, so the reduction of  $h = \chi_{\mathfrak{p}}$  modulo  $\mathfrak{p}$  is reducible in  $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}[x]$ , a contradiction.  $\square$

We derive a few interesting corollaries.

**Corollary 2.5.** *Let  $B$  be a central simple algebra over a number field  $K$  of dimension  $n^2$ , and  $L$  a degree  $n$  field extension of  $K$  which embeds into  $B$ . Let  $\mathfrak{p}$  be a prime of  $K$  which splits in  $B$  and is unramified in  $L$ , and let  $\mathcal{E}_{\mathfrak{p}}$  a maximal order in  $B_{\mathfrak{p}}$ . Then the following are equivalent:*

- (1)  $\mathcal{O}_L \subset \mathcal{E}_{\mathfrak{p}}$
- (2)  $\mathcal{O}_{L_{\mathfrak{P}}} \subset \mathcal{E}_{\mathfrak{p}}$  for some prime  $\mathfrak{P}$  of  $L$  dividing  $\mathfrak{p}\mathcal{O}_L$ .
- (3)  $\bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{L_{\mathfrak{P}}} \subset \mathcal{E}_{\mathfrak{p}}$  where the sum is over all primes  $\mathfrak{P} \mid \mathfrak{p}\mathcal{O}_L$ .

*Proof.* The implications (3) implies (2) implies (1) are obvious. That (1) implies (3) is exactly as in the proof of Theorem 2.1 using Lemma 2.4: If  $S$  is the integral closure in  $L$  of  $(\mathcal{O}_K)_{\mathfrak{p}}$  (the localization of  $\mathcal{O}_K$  at the prime  $\mathfrak{p}$ ), then Lemma 2.4 shows that  $\mathcal{O}_L \subset \mathcal{E}_{\mathfrak{p}}$  implies that  $S \subset \mathcal{E}_{\mathfrak{p}}$ . Thus  $S \otimes_{(\mathcal{O}_K)_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}} \subset \mathcal{E}_{\mathfrak{p}}$ . But by Proposition II.4 of [20],  $S \otimes_{(\mathcal{O}_K)_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}} \cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{L_{\mathfrak{P}}}$ , so  $\bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{L_{\mathfrak{P}}} \subset \mathcal{E}_{\mathfrak{p}}$ .  $\square$



It is clear from Theorem 2.1, that when  $\mathcal{O}_L \subset \mathcal{R}$  and a prime  $\mathfrak{p}$  of  $K$  splits completely in  $L$ , then  $\mathcal{O}_L$  is contained in every maximal order in an apartment of the  $SL_n(K_{\mathfrak{p}})$  building which contains  $\mathcal{R}_{\mathfrak{p}}$ . In particular  $\mathcal{O}_L$  is contained in every vertex of a fundamental chamber containing  $\mathcal{R}_{\mathfrak{p}}$  (corresponding to the homothety class  $[0, \dots, 0]$ ). On the other hand, if  $\mathfrak{p}$  is inert in  $L$ , then similarly, we see from Corollary 2.2 that  $\mathcal{O}_L$  is contained in precisely one vertex of the fundamental chamber; indeed it is contained in a unique vertex in the building. This last observation can be cast in purely local terms.

**Corollary 2.6.** *Let  $F$  be a nondyadic nonarchimedean local field, and  $E$  the degree  $n \geq 3$  unramified extension of  $F$ . Then there is a unique maximal order  $\mathcal{E}$  of  $M_n(F)$  containing  $\mathcal{O}_E$ .*

*Proof.* Let  $K$  be a number field with a finite prime  $\nu$  so that  $F = K_{\nu}$ , and let  $B$  be a central simple algebra of dimension  $n^2$  over  $K$  which is split at  $\nu$ , that is the local index  $m_{\nu} = 1$ . Let  $m_{\mathfrak{p}_i}$ ,  $i = 1, \dots, s$  be the nontrivial local indices associated to  $B$ . By the Grunwald-Wang theorem [3], we may choose a cyclic extension  $L/K$  of degree  $n$  so that  $m_{\mathfrak{p}_i} = [L_{\mathfrak{p}_i} : K_{\mathfrak{p}_i}]$  for all  $\mathfrak{p}_i \mid \nu$ ,  $i = 1, 2, \dots, s$ , and  $[L_{\mathfrak{p}} : K_{\mathfrak{p}}] = n$ . Since  $\nu \nmid 2$  we may choose the extension so that  $L_{\mathfrak{p}}/K_{\mathfrak{p}}$  is unramified which means that  $\nu$  is inert in  $L$ , so that  $E = L_{\mathfrak{p}}$ . The conditions on the local indices guarantee that  $L$  embeds in  $B$  via the Albert-Brauer-Hasse-Noether theorem. So we assume  $K \subset L \subset B$ , and choose a maximal order  $\mathcal{R}$  containing  $\mathcal{O}_L$ . Then  $\mathcal{O}_L \subset \mathcal{R}_{\nu}$ . By the previous corollary,  $\mathcal{O}_{L_{\mathfrak{p}}} \subset \mathcal{R}_{\nu}$  for the unique prime  $\mathfrak{p}$  lying above  $\nu$ . We claim that this is the unique maximal order of  $\mathcal{B}_{\nu} = M_n(K_{\nu})$  containing  $\mathcal{O}_{L_{\mathfrak{p}}}$ . Indeed since  $\nu$  is inert in  $L$ , Theorem 2.1 characterizes the homothety classes of lattices which correspond to the maximal orders containing  $\mathcal{O}_{L_{\mathfrak{p}}}$ , as  $[\ell, \dots, \ell] = [0, \dots, 0]$ , the class corresponding only to  $\mathcal{R}_{\nu}$ .  $\square$

### 3. GLOBAL EMBEDDINGS

In this section we apply Theorem 2.1 to the problem of determining which isomorphism classes of maximal orders in the global algebra  $B$  admit an embedding of  $\mathcal{O}_L$ .

To begin we need to parametrize the isomorphism classes of maximal orders in  $B$ . Recall we have fixed a maximal order  $\mathcal{R}$  which contains  $\mathcal{O}_L$ , and now need to give a set of representatives of the distinct isomorphism classes comprising the genus of maximal orders.

**3.1. Parametrizing the Genus.** We want to fix a convenient parametrization of the isomorphism classes of maximal orders in  $B$  which is compatible with the local embedding theory of the previous section. We begin by defining a class field  $K(\mathcal{R})$  associated to the maximal order  $\mathcal{R}$ . We refer to [16] for most of the details.

Given a maximal order  $\mathcal{R} \subset B$ , and a prime  $\nu$  of  $K$ , we define localizations  $\mathcal{R}_{\nu} \subseteq B_{\nu}$  by:

$$\mathcal{R}_\nu = \begin{cases} \mathcal{R} \otimes_{\mathcal{O}} \mathcal{O}_\nu & \text{if } \nu \text{ is finite} \\ \mathcal{R} \otimes_{\mathcal{O}} K_\nu = B_\nu & \text{if } \nu \text{ is infinite} \end{cases}$$

For every finite prime  $\nu$ , it is well-known that  $\mathcal{R}_\nu$  is a maximal order of  $B_\nu$ . Let  $\mathcal{N}(\mathcal{R}_\nu)$  denote the normalizer of  $\mathcal{R}_\nu$  in  $B_\nu^\times$ , and  $nr(\mathcal{N}(\mathcal{R}_\nu))$  its reduced norm in  $K_\nu^\times$ . When  $\nu$  is an infinite prime,  $\mathcal{N}(\mathcal{R}_\nu) = B_\nu^\times$ . If  $\nu$  is finite, it is convenient to distinguish three cases: If  $\nu$  splits in  $B$ , then  $B_\nu \cong M_n(K_\nu)$  and every maximal order is conjugate by an element of  $B_\nu^\times$  to  $M_n(\mathcal{O}_\nu)$ , so every normalizer is conjugate to  $GL_n(\mathcal{O}_\nu)K_\nu^\times$  (37.26 of [19]), while if  $\nu$  totally ramified in  $B$ ,  $\mathcal{R}_\nu$  is the unique maximal order of the division algebra  $B_\nu$  [19], so  $\mathcal{N}(\mathcal{R}_\nu) = B_\nu^\times$ . For  $\nu$  partially ramified in  $B$ ,  $B_\nu \cong M_{\kappa_\nu}(D_\nu)$  with  $D_\nu$  a division algebra of degree  $m_\nu$  over  $K_\nu$ , so  $\mathcal{R}_\nu$  is conjugate to  $M_{\kappa_\nu}(\Lambda_\nu)$  where  $\Lambda_\nu$  is the unique maximal order of  $D_\nu$  (17.3 of [19]). If we choose  $u \in \Lambda_\nu^\times$  then a conjugate of  $U = \text{diag}(u, 1, \dots, 1) \in \mathcal{N}(\mathcal{R}_\nu)$ , and  $nr(U) = nr(u) \in nr(\mathcal{N}(\mathcal{R}_\nu))$ . Finally, we recall (see p145 of [19]) that  $\Lambda_\nu$  contains the ring of integers of an unramified cyclic extension of  $K_\nu$ .

It follows (p 153 of [19]) that for  $\nu$  infinite or totally ramified,  $nr(\mathcal{N}(\mathcal{R}_\nu)) = nr(B_\nu^\times) = K_\nu^\times$ , and in all other cases  $\mathcal{O}_\nu^\times (K_\nu^\times)^n \subseteq nr(\mathcal{N}(\mathcal{R}_\nu))$ , with equality when  $\nu$  is split. Thus in all cases  $\mathcal{O}_\nu^\times \subset nr(\mathcal{N}(\mathcal{R}_\nu))$  and for  $\nu$  infinite or totally ramified in  $B$ ,  $K_\nu^\times = nr(\mathcal{N}(\mathcal{R}_\nu))$ , a fact that will be important in associating a class field to  $\mathcal{R}$ .

The type number of  $\mathcal{R}$  is the cardinality of the double coset space  $B^\times \backslash J_B / \mathfrak{N}(\mathcal{R})$  where  $J_B$  are the ideles of  $B$ . To make use of class field theory, we need to realize this quotient in terms of the arithmetic of  $K$ . Let  $J_K$  denote the idele group of  $K$ .

**Theorem 3.1.** *The reduced norm on  $B$  induces a bijection*

$$nr : B^\times \backslash J_B / \mathfrak{N}(\mathcal{R}) \rightarrow K^\times \backslash J_K / nr(\mathfrak{N}(\mathcal{R})).$$

*The group  $K^\times \backslash J_K / nr(\mathfrak{N}(\mathcal{R}))$  is abelian with exponent  $n$ .*

*Proof.* Here of course we also use  $nr$  as the induced reduced norm map on ideles. The proof is exactly as in Theorems 3.1 and 3.2 of [16].  $\square$

We have seen above that the distinct isomorphism classes of maximal orders in  $B$  are in one-to-one correspondence with the double cosets in the group  $G = K^\times \backslash J_K / nr(\mathfrak{N}(\mathcal{R}))$ . Put  $H_{\mathcal{R}} = K^\times nr(\mathfrak{N}(\mathcal{R}))$  and  $G_{\mathcal{R}} = J_K / H_{\mathcal{R}}$ . Since  $J_K$  is abelian,  $G$  and  $G_{\mathcal{R}}$  are naturally isomorphic, and since  $H_{\mathcal{R}}$  contains a neighborhood of the identity in  $J_K$ , it is an open subgroup (Proposition II.6 of [13]).

Since  $H_{\mathcal{R}}$  is an open subgroup of  $J_K$  having finite index, there is by class field theory [14], a class field,  $K(\mathcal{R})$ , associated to it. The extension  $K(\mathcal{R})/K$  is an abelian extension with  $Gal(K(\mathcal{R})/K) \cong G_{\mathcal{R}} = J_K / H_{\mathcal{R}}$  and with  $H_{\mathcal{R}} = K^\times N_{K(\mathcal{R})/K}(J_{K(\mathcal{R})})$ . Moreover, a prime  $\nu$  of  $K$  (possibly infinite) is unramified in  $K(\mathcal{R})$  if and only if  $\mathcal{O}_\nu^\times \subset H_{\mathcal{R}}$ , and splits completely if and only if  $K_\nu^\times \subset H_{\mathcal{R}}$ . Here if  $\nu$  is archimedean, we take  $\mathcal{O}_\nu^\times = K_\nu^\times$ . From our

computations just above, we saw that  $\mathcal{O}_\nu^\times$  is always contained in  $H_{\mathcal{R}}$ . In particular  $K(\mathcal{R})/K$  is an everywhere unramified abelian extension of  $K$ .

The Galois group  $G = \text{Gal}(K(\mathcal{R})/K)$  is a finite abelian group of exponent  $n$ . We wish to specify a set of generators for the group as Artin symbols,  $(\nu, K(\mathcal{R})/K)$ , in such a way that we can control the splitting behavior of  $\nu$  in the field  $L$ . As  $L$  is an arbitrary extension of  $K$  of degree  $n$ , this requires some care.

We have assumed that  $L \subset B$ . Put  $L_0 = K(\mathcal{R}) \cap L$  and  $\widehat{L}_0 = \widehat{L} \cap K(\mathcal{R})$  where  $\widehat{L}$  is the Galois closure of  $L$ . Then  $L_0 \subset \widehat{L}_0$  and we define subgroups of  $G$ :  $\widehat{H} = \text{Gal}(K(\mathcal{R})/\widehat{L}_0) \subseteq H = \text{Gal}(K(\mathcal{R})/L_0)$ . We write the finite abelian groups  $\widehat{H}$ ,  $H/\widehat{H}$ , and  $G/H$  as a direct product of cyclic groups:

$$(6) \quad G/H = \langle \rho_1 H \rangle \times \cdots \times \langle \rho_r H \rangle,$$

$$(7) \quad H/\widehat{H} = \langle \sigma_1 \widehat{H} \rangle \times \cdots \times \langle \sigma_s \widehat{H} \rangle,$$

$$(8) \quad \widehat{H} = \langle \tau_1 \rangle \times \cdots \times \langle \tau_t \rangle.$$

The following proposition is clear.

**Proposition 3.2.** *Every element  $\varphi \in G$  can be written uniquely as  $\varphi = \rho_1^{a_1} \cdots \rho_r^{a_r} \sigma_1^{b_1} \cdots \sigma_s^{b_s} \tau_1^{c_1} \cdots \tau_t^{c_t}$  where  $0 \leq a_i < |\rho_i H|$ ,  $0 \leq b_j < |\sigma_j \widehat{H}|$ , and  $0 \leq c_k < |\tau_k|$ , with  $|\cdot|$  the order of the element in the respective group.*

Next we characterize each of these generators in terms of Artin symbols. Since the vehicle to accomplish this is the Chebotarev density theorem which provides an infinite number of choices for primes, we may and do assume without loss that the primes we choose to define the Artin symbols are unramified in both  $\widehat{L}$  and  $B$ .

First consider the elements  $\tau_k \in \widehat{H} = \text{Gal}(K(\mathcal{R})/\widehat{L}_0)$ . By Lemma 7.14 of [18], there exist infinitely many primes  $\nu_k$  of  $K$  so that  $\tau_k = (\nu_k, K(\mathcal{R})/K)$  and for which there exists a prime  $Q_k$  of  $\widehat{L}$  with inertia degree  $f(Q_k | \nu_k) = 1$ . Since  $\widehat{L}/K$  is Galois (and the prime  $\nu_k$  is unramified by assumption), this implies  $\nu_k$  splits completely in  $\widehat{L}$ , hence also in  $L$ .

Next consider  $\sigma_j \widehat{H}$  with  $\sigma_j \in H = \text{Gal}(K(\mathcal{R})/L_0)$ . Again by Lemma 7.14 of [18], there exist infinitely many primes  $\mu_j$  of  $K$  so that  $\sigma_j = (\mu_j, K(\mathcal{R})/K)$  and for which there exists a prime  $Q_j$  of  $L$  with inertia degree  $f(Q_j | \mu_j) = 1$ .

Finally consider  $\rho_k H$  with  $\rho_k \in G = \text{Gal}(K(\mathcal{R})/K)$ . By Chebotarev, there exist infinitely many primes  $\lambda_i$  of  $K$  so that  $\rho_i = (\lambda_i, K(\mathcal{R})/K)$ . For later convenience, we note that by standard properties of the Artin symbol,  $\overline{\rho}_i = \rho_i|_{L_0} = (\lambda_i, L_0/K)$  whose order in  $\text{Gal}(L_0/K)$  is equal to the inertia degree  $f(\lambda_i; L_0/K)$ .

As we said above, we have assumed without loss that all the primes  $\lambda_i, \mu_j, \nu_k$  are unramified in  $\widehat{L}$  and  $B$ .

**3.2. Fixing representatives of the isomorphism classes.** As above,  $\mathcal{R}$  is a fixed maximal order of  $B$  containing  $\mathcal{O}_L$ . For a finite prime  $\mathfrak{p}$  of  $K$  which splits in  $B$ , we recall the setup and notation of section 2 where we fixed an apartment in the affine building for  $SL_n(K_{\mathfrak{p}})$  which contains the vertex  $\mathcal{R}_{\mathfrak{p}} \subset B_{\mathfrak{p}} \cong M_n(K_{\mathfrak{p}})$ .

We are interested in vertices of the form  $\mathcal{R}(k, \ell)$  defined in equation (3). Because we shall vary the prime  $\mathfrak{p}$  in the parametrization below, we will write  $\mathcal{R}_{\mathfrak{p}}(k, \ell)$  for  $\mathcal{R}(k, \ell)$  to make the dependence on  $\mathfrak{p}$  explicit. Recall that  $\mathcal{R}_{\mathfrak{p}}(k, \ell)$  corresponds to the homothety class  $[\underbrace{\ell, \dots, \ell}_k, 0, \dots, 0] \in \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$  which has type  $k\ell \pmod{n}$ .

Now for the primes  $\lambda_i$ ,  $\mu_j$ , and  $\nu_k$  we specified above to parametrize  $G = \text{Gal}(K(\mathcal{R})/K)$ , fix the following local orders using the decomposition of  $G$  into  $G/H$ ,  $H/\widehat{H}$ , and  $\widehat{H}$ :

For each prime  $\lambda_i$  ( $i = 1, \dots, r$ ) whose Artin symbol  $(\lambda_i, K(\mathcal{R})/K) = \rho_i$  gives one of the generators  $\rho_i H$  of  $G/H$ , we fix vertices  $\mathcal{R}_{\lambda_i}(m, 1)$ ,  $m = 0, 1, \dots, |\rho_i H| - 1$  with associated homothety classes  $[0, \dots, 0]$ ,  $[1, 0, \dots, 0]$ ,  $[1, 1, 0, \dots, 0]$ ,  $\dots$ ,  $[\underbrace{1, \dots, 1}_{|\rho_i H| - 1}, 0, \dots, 0]$ .

For each prime  $\mu_j$  ( $j = 1, \dots, s$ ) whose Artin symbol  $(\mu_j, K(\mathcal{R})/K) = \sigma_j$  gives one of the generators  $\sigma_j \widehat{H}$  of  $H/\widehat{H}$ , we fix vertices  $\mathcal{R}_{\mu_j}(1, m)$ ,  $m = 0, 1, \dots, |\sigma_j \widehat{H}| - 1$  with associated homothety classes  $[0, \dots, 0]$ ,  $[1, 0, \dots, 0]$ ,  $[2, 0, \dots, 0]$ ,  $\dots$ ,  $[|\sigma_j \widehat{H}| - 1, 0, \dots, 0]$ . Recall that each  $\mu_j$  factors into primes of  $L$  with at least one having inertia degree one over  $\mu_j$ . Since the representation above depends on the order in which we list those primes (see equation (2)), we assume the first prime has degree one.

Finally, for each prime  $\nu_k$  ( $k = 1, \dots, t$ ) whose Artin symbol  $(\nu_k, K(\mathcal{R})/K) = \tau_k$  is one of the generators of  $\widehat{H}$ , we fix vertices  $\mathcal{R}_{\nu_k}(m, 1)$ ,  $m = 0, 1, \dots, |\tau_k| - 1$  with associated homothety classes  $[0, \dots, 0]$ ,  $[1, 0, \dots, 0]$ ,  $[1, 1, 0, \dots, 0]$ ,  $\dots$ ,  $[\underbrace{1, \dots, 1}_{|\tau_k| - 1}, 0, \dots, 0]$ . Again recall

that each prime  $\nu_k$  splits completely in  $L$ , so all the inertia degrees are one.

*Remark 3.3.* We note that using Theorem 2.1 and the conventions on the degree one primes listed above, we see that  $\mathcal{O}_L$  is a subset of  $\mathcal{R}_{\mu_j}(1, m)$  for every value of  $m$ , and of  $\mathcal{R}_{\nu_k}(m', 1)$  for  $0 \leq m' \leq n$ .

Now we use the local-global correspondence for orders to define global orders from the above local factors. Fix the following notation:

$$\begin{aligned} \mathbf{a} &= (a_i) \in \mathbb{Z}/|\rho_1 H|\mathbb{Z} \times \dots \times \mathbb{Z}/|\rho_r H|\mathbb{Z}, \\ \mathbf{b} &= (b_j) \in \mathbb{Z}/|\sigma_1 \widehat{H}|\mathbb{Z} \times \dots \times \mathbb{Z}/|\sigma_s \widehat{H}|\mathbb{Z}, \\ \mathbf{c} &= (c_k) \in \mathbb{Z}/|\tau_1|\mathbb{Z} \times \dots \times \mathbb{Z}/|\tau_t|\mathbb{Z}. \end{aligned}$$

Here we assume the coordinates  $a_i, b_j, c_k$  are integers which are the least non-negative residues corresponding to the moduli. Define maximal orders in  $B$  via the local-global correspondence:

$$\mathcal{D}_{\mathfrak{p}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}} = \begin{cases} \mathcal{R}_{\mathfrak{p}} & \text{if } \mathfrak{p} \notin \{\lambda_i, \mu_j, \nu_k\}, \\ \mathcal{R}_{\lambda_i}(a_i, 1) & \text{if } \mathfrak{p} = \lambda_i, i = 1, \dots, r, \\ \mathcal{R}_{\mu_j}(1, b_j) & \text{if } \mathfrak{p} = \mu_j, j = 1, \dots, s, \\ \mathcal{R}_{\nu_k}(c_k, 1) & \text{if } \mathfrak{p} = \nu_k, k = 1, \dots, t. \end{cases}$$

We claim that such a collection of maximal orders parametrizes the isomorphism classes of maximal orders in  $B$ . That is, given any maximal order  $\mathcal{E}$  in  $B$ , we show there are unique tuples  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  so that  $\mathcal{E} \cong \mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$ . To see this let  $\mathfrak{M}$  denote the set of all maximal orders in  $B$ , and define a map (called the  $G_{\mathcal{R}}$ -valued distance idele)  $\delta : \mathfrak{M} \times \mathfrak{M} \rightarrow G_{\mathcal{R}} = J_K/H_{\mathcal{R}}$  (where  $H_{\mathcal{R}} = K^{\times} nr(\mathfrak{N}(\mathcal{R}))$ ) as follows: Let  $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{M}$ . For a finite prime  $\mathfrak{p}$  of  $K$  which splits in  $B$ , we have the notion of the type distance between their localizations:  $td_{\mathfrak{p}}(\mathcal{R}_{1\mathfrak{p}}, \mathcal{R}_{2\mathfrak{p}}) \in \mathbb{Z}/n\mathbb{Z}$  (see section 2 of [16]). If  $\mathfrak{p}$  is either archimedean or finite and ramified in  $B$ , define  $td_{\mathfrak{p}}(\mathcal{R}_{1\mathfrak{p}}, \mathcal{R}_{2\mathfrak{p}}) = 0$ . Recall that since  $\mathcal{R}_{1\mathfrak{p}} = \mathcal{R}_{2\mathfrak{p}}$  for almost all  $\mathfrak{p}$ ,  $td_{\mathfrak{p}}(\mathcal{R}_{1\mathfrak{p}}, \mathcal{R}_{2\mathfrak{p}}) = 0$  for almost all primes  $\mathfrak{p}$ . Let  $\delta(\mathcal{R}_1, \mathcal{R}_2)$  be the image in  $G_{\mathcal{R}}$  of the idele  $(\pi_{\mathfrak{p}}^{td_{\mathfrak{p}}(\mathcal{R}_{1\mathfrak{p}}, \mathcal{R}_{2\mathfrak{p}})})$ , where  $\pi_{\mathfrak{p}}$  is a fixed uniformizing parameter in  $K_{\mathfrak{p}}$ . Note that while the idele is not well-defined, its image in  $G_{\mathcal{R}}$  is since the local factor at the finite split primes has the form  $K_{\mathfrak{p}}^{\times}/\mathcal{O}_{\mathfrak{p}}^{\times}(K_{\mathfrak{p}}^{\times})^n$ .

We now show that the orders  $\{\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}\}$  parametrize the isomorphism classes of maximal orders in  $B$ .

**Proposition 3.4.** *Let  $\mathcal{R}$  be a fixed maximal order in  $B$ , and consider the collection of maximal orders  $\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  defined above.*

- (1) *If  $\mathcal{E}$  is a maximal order in  $B$  and  $\mathcal{E} \cong \mathcal{R}$ , then  $\delta(\mathcal{R}, \mathcal{E})$  is trivial.*
- (2) *If  $\mathcal{E} \cong \mathcal{E}'$  are maximal orders in  $B$ , then  $\delta(\mathcal{R}, \mathcal{E}) = \delta(\mathcal{R}, \mathcal{E}')$ .*
- (3)  *$\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}} \cong \mathcal{D}^{\mathbf{a}', \mathbf{b}', \mathbf{c}'}$  if and only if  $\mathbf{a} = \mathbf{a}'$ ,  $\mathbf{b} = \mathbf{b}'$ , and  $\mathbf{c} = \mathbf{c}'$ .*

*Proof.* The first two statements are proven exactly as in Proposition 3.3 of [16]. For the last, let  $\mathfrak{q}$  be a finite prime of  $K$  and  $\pi_{\mathfrak{q}}$  the corresponding uniformizing parameter. Let  $e_{\mathfrak{q}}$  denote the idele with  $\pi_{\mathfrak{q}}$  in the  $\mathfrak{q}$ th place and 1's elsewhere. Observe that Artin reciprocity identifies the image of  $e_{\mathfrak{q}}$  in  $G_R = J_K/H_{\mathcal{R}}$  with the Artin symbol  $(\mathfrak{q}, K(\mathcal{R})/K)$ . It follows that

$$\delta(\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}, \mathcal{D}^{\mathbf{a}', \mathbf{b}', \mathbf{c}'}) \leftrightarrow \prod_{i=1}^g \rho_i^{a'_i - a_i} \prod_{j=1}^s \sigma_j^{b'_j - b_j} \prod_{k=1}^t \tau_k^{c'_k - c_k} \in \text{Gal}(K(\mathcal{R})/K),$$

which is trivial if and only if  $\mathbf{a} = \mathbf{a}'$ ,  $\mathbf{b} = \mathbf{b}'$ , and  $\mathbf{c} = \mathbf{c}'$  by Proposition 3.2. The result is now immediate.  $\square$

4. EMBEDDING  $\mathcal{O}_L$  INTO MAXIMAL ORDERS IN  $B$ 

Pivotal to understanding which maximal orders in  $B$  admit an embedding of  $\mathcal{O}_L$  is the following characterization.

**Theorem 4.1.** *Assume that  $\mathcal{O}_L \subset \mathcal{R} \subset B$ . Let  $\mathcal{E}$  be another maximal order in  $B$ , and let  $\delta(\mathcal{R}, \mathcal{E})$  denote the distance idele defined above. Then  $\mathcal{E}$  admits an embedding of  $\mathcal{O}_L$  if and only if the idelic Artin symbol  $(\delta(\mathcal{R}, \mathcal{E}), L_0/K)$  is trivial in  $\text{Gal}(L_0/K)$ .*

*Proof.* Assume that  $\mathcal{E}$  admits an embedding of  $\mathcal{O}_L$ ; that is,  $\mathcal{E}$  contains a conjugate of  $\mathcal{O}_L$ . By Proposition 3.4, we may replace  $\mathcal{E}$  by a conjugate order without changing the distance idele, so we may assume without loss that  $\mathcal{O}_L \subset \mathcal{R} \cap \mathcal{E}$ . The distance idele  $\delta(\mathcal{R}, \mathcal{E})$  is the coset  $\tilde{\delta}H_{\mathcal{R}} \in G_{\mathcal{R}}$  where  $\tilde{\delta} = \prod_{\mathfrak{p}} e_{\mathfrak{p}}^{td_{\mathfrak{p}}(\mathcal{R}_{\mathfrak{p}}, \mathcal{E}_{\mathfrak{p}})}$  with the product over all the finite primes of  $K$  which split in  $B$  and for which  $\mathcal{R}_{\mathfrak{p}} \neq \mathcal{E}_{\mathfrak{p}}$ , and with  $e_{\mathfrak{p}} = (1, \dots, 1, \pi_{\mathfrak{p}}, 1, \dots, 1) \in J_K$ . Write  $\sigma = (\delta(\mathcal{R}, \mathcal{E}), L_0/K) \in \text{Gal}(L_0/K)$ . Then  $\sigma = \prod_{\mathfrak{p}} (\mathfrak{p}, L_0/K)^{td_{\mathfrak{p}}(\mathcal{R}_{\mathfrak{p}}, \mathcal{E}_{\mathfrak{p}})}$  where  $(\mathfrak{p}, L_0/K)$  is the Artin symbol. Since  $\mathcal{O}_L \subset \mathcal{E}_{\mathfrak{p}}$ , we know by Theorem 2.1 that  $\mathcal{E}_{\mathfrak{p}}$  corresponds to the homothety class  $[\underbrace{\ell_1, \dots, \ell_1}_{f_1}, \underbrace{\ell_2, \dots, \ell_2}_{f_2}, \dots, \underbrace{\ell_g, \dots, \ell_g}_{f_g}]$  for integers  $\ell_i$  and where the  $f_i$  are the

inertia degrees of the primes of  $L$  lying above  $\mathfrak{p}$ . Thus,  $td_{\mathfrak{p}}(\mathcal{R}_{\mathfrak{p}}, \mathcal{E}_{\mathfrak{p}}) \equiv \sum_{i=1}^g \ell_i f_i \pmod{n}$ . Now each  $f_i$  is divisible by  $f_0 = f(\mathfrak{p}; L_0/K)$ , the inertia  $\mathfrak{p}$  gains up to  $L_0$  (recall  $L_0/K$  is Galois). But  $f_0$  is order of the Artin symbol  $(\mathfrak{p}; L_0/K)$ , so each term in the product is trivial, hence so is  $\sigma$ .

For the converse, we suppose that  $\sigma = (\delta(\mathcal{R}, \mathcal{E}), L_0/K)$  is the trivial element of  $\text{Gal}(L_0/K)$  and show that  $\mathcal{E}$  admits an embedding of  $\mathcal{O}_L$ . In parametrizing the isomorphism classes of maximal orders of  $B$ , we wrote  $\text{Gal}(L_0/K)$  as a product of cyclic groups: With  $G = \text{Gal}(K(\mathcal{R})/K)$  and  $H = \text{Gal}(K(\mathcal{R})/L_0)$  we have  $G/H \cong \langle \rho_1 H \rangle \times \dots \times \langle \rho_r H \rangle$ , so  $\text{Gal}(L_0/K) \cong \langle \bar{\rho}_1 \rangle \times \dots \times \langle \bar{\rho}_r \rangle$  where  $\bar{\rho}_i$  is the restriction of  $\rho_i$  to  $L_0$ . Each  $\rho_i$  is the Artin symbol  $(\lambda_i, K(\mathcal{R})/K)$ .

Our given  $\mathcal{E}$  is isomorphic to a unique element,  $\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$ , of our parametrizing set of representatives of the isomorphism classes of maximal orders, and since  $\delta(\mathcal{R}, \mathcal{E}) = \delta(\mathcal{R}, \mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}})$  by Proposition 3.4, it is sufficient to show that  $\sigma = (\delta(\mathcal{R}, \mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}), L_0/K)$  trivial in  $\text{Gal}(L_0/K)$  implies that  $\mathcal{O}_L \subset \mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$ .

We are assuming that  $\mathcal{O}_L \subset \mathcal{R}$ , and since  $\mathcal{R}$  and  $\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  are equal at all primes outside the finite set  $T = \{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s, \nu_1, \dots, \nu_t\}$  of parametrizing primes,  $\mathcal{O}_L \subset \mathcal{D}_{\mathfrak{p}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  for all primes  $\mathfrak{p} \notin T$ . For  $\mathfrak{p} = \mu_j$  or  $\nu_k$ , we have  $\mathcal{O}_L \subset \mathcal{D}_{\mathfrak{p}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  by Remark 3.3. So we need to deal only with the primes  $\lambda_i$ .

Recall that for  $\mathfrak{p} \notin T$  we have  $td_{\mathfrak{p}}(\mathcal{R}_{\mathfrak{p}}, \mathcal{D}_{\mathfrak{p}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}) = 0$  so that  $(\delta(\mathcal{R}, \mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}), L_0/K) = \prod_{\mathfrak{p} \in T} (\mathfrak{p}, L_0/K)^{td_{\mathfrak{p}}(\mathcal{R}_{\mathfrak{p}}, \mathcal{D}_{\mathfrak{p}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}})}$ . Furthermore the primes  $\mu_j, \nu_k \in T$  correspond to elements in  $\text{Gal}(K(\mathcal{R})/L_0)$  so their restriction to  $L_0$  is trivial. Thus our condition reduces to  $\prod_{i=1}^r (\lambda_i, L_0/K)^{td_{\lambda_i}(\mathcal{R}_{\lambda_i}, \mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}})} = 1$ . Since  $\text{Gal}(L_0/K)$  is the direct product

$\prod_{i=1}^r \langle (\lambda_i, L_0/K) \rangle$ , we have that  $\prod_{i=1}^r (\lambda_i, L_0/K)^{td_{\lambda_i}(\mathcal{R}_{\lambda_i}, \mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}})} = 1$  implies each of the terms  $(\lambda_i, L_0/K)^{td_{\lambda_i}(\mathcal{R}_{\lambda_i}, \mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}})} = 1$ , so  $td_{\lambda_i}(\mathcal{R}_{\lambda_i}, \mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}) \equiv 0 \pmod{f(\lambda_i; L_0/K)}$  since  $f(\lambda_i; L_0/K)$  is the order of the Artin symbol. Since we have parametrized  $\mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  with maximal orders whose corresponding homothety classes have types  $0, 1, \dots, f(\lambda_i; L_0/K) - 1$ , we have that  $\mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}} = \mathcal{R}_{\lambda_i}$ , so  $\mathcal{O}_L \subset \mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$ , and hence  $\mathcal{O}_L \subset \mathcal{D}_{\mathfrak{p}}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  for all primes  $\mathfrak{p}$  which completes the proof.  $\square$

**Corollary 4.2.** *The ratio of the number of isomorphism classes of maximal orders in  $B$  which admit an embedding of  $\mathcal{O}_L$  to the total number of isomorphism classes of maximal orders is  $[L_0 : K]^{-1}$  where  $L_0 = K(\mathcal{R}) \cap L$ .*

*Proof.* Let  $\{\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}\}$  be the parametrization of the isomorphism classes of maximal orders of  $B$  given in section 2. We must show that exactly  $[K(\mathcal{R}) : L_0]$  of these orders admit an embedding of  $\mathcal{O}_L$ . By the proof of Theorem 4.1, we have that  $\sigma = (\delta(\mathcal{R}, \mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}), L_0/K) = \prod_{i=1}^r (\lambda_i, L_0/K) = 1$  in  $\text{Gal}(L_0/K)$  if and only if  $\mathcal{D}_{\lambda_i}^{\mathbf{a}, \mathbf{b}, \mathbf{c}} = \mathcal{R}_{\lambda_i}$  for  $i = 1, \dots, r$ ; or equivalently,  $\mathbf{a} = 0$ . But the number of orders  $\{\mathcal{D}^{\mathbf{a}, \mathbf{b}, \mathbf{c}}\}$  with  $\mathbf{a} = 0$  is  $|H/\widehat{H}||\widehat{H}| = |H| = [K(\mathcal{R}) : L_0]$ , which finishes the proof.  $\square$

## 5. EXAMPLES AND APPLICATIONS

Corollary 4.2 allows a significant refinement of the contexts in which selectivity can occur. We begin with the first instance in which selectivity was noted, by recovering Chevalley's [9] elegant result when  $B = M_n(K)$ .

**Corollary 5.1** (Chevalley). *The ratio of the number of isomorphism classes of maximal orders in  $B = M_n(K)$  into which  $\mathcal{O}_L$  can be embedded to the total number of isomorphism classes of maximal orders is  $[\widetilde{K} \cap L : K]^{-1}$  where  $\widetilde{K}$  is the Hilbert class field of  $K$ .*

*Proof.* We are assuming that  $\mathcal{O}_L \subset \mathcal{R}$ , where  $\mathcal{R}$  our fixed maximal order in  $B$ . We need only show that  $\widetilde{K} \cap L = K(\mathcal{R}) \cap L$ .

When  $B = M_n(K)$ , we have that  $H_{\mathcal{R}} = K^{\times} J_K^n J_{K, S_{\infty}}$  so that  $G_{\mathcal{R}} = C_K / C_K^n$  where  $C_K$  is the ideal class group of  $K$ . Since  $G_{\mathcal{R}}$  is a quotient of the ideal class group (i.e.,  $\text{Gal}(\widetilde{K}/K)$ ),  $K(\mathcal{R}) \subset \widetilde{K}$  is the subfield corresponding to  $C_K^n$ . In particular  $K(\mathcal{R}) \cap L = L_0 \subset \widetilde{K} \cap L$ .

Conversely, since  $K(\mathcal{R})$  corresponds to  $C_K^n$ , it is the maximal abelian unramified extension of  $K$  with exponent  $n$ . Since  $[L : K] = n$ ,  $\widetilde{K} \cap L$  is an unramified extension of  $K$  having exponent  $n$ , so by maximality  $\widetilde{K} \cap L \subset K(\mathcal{R}) \cap L$ .  $\square$

Examples of selective and non-selective orders in matrix algebras are given in [16]. Also in that paper, we prove that orders in central simple division algebras of (odd) prime degree are never selective. We now show exactly how this generalizes to arbitrary degree  $n \geq 3$ .

Recall that locally there is an isomorphism  $B_\nu \cong M_{\kappa_\nu}(D_\nu)$ , where  $D_\nu$  is a central simple division algebra of dimensions  $m_\nu^2$  over  $K_\nu$ , and we have  $n = \kappa_\nu m_\nu$ . We recall the Albert-Brauer-Hasse-Noether theorem:

**Theorem.** (ABNH) *Let the notation be as above, and suppose that  $[L : K] = n$ . Then there is an embedding of  $L/K$  into  $B$  if and only if for each prime  $\nu$  of  $K$  and for all primes  $\mathfrak{P}$  of  $L$  lying above  $\nu$ ,  $m_\nu \mid [L_\mathfrak{P} : K_\nu]$ .*

The first case we consider is a non-split algebra  $B$  which is fully ramified at some prime (necessarily a finite prime since  $n > 2$ ). We show that in this case there is never selectivity.

**Corollary 5.2.** *Suppose there is a prime  $\nu$  of  $K$  so that  $B_\nu$  is a division algebra, i.e.,  $m_\nu = n$  and  $L \subset B$ . Let  $\Omega \subseteq \mathcal{O}_L$  be any  $\mathcal{O}_K$ -order. Then  $\Omega$  is never selective; that is, every maximal order in  $B$  admits an embedding of  $\Omega$ .*

*Proof.* It is enough to show that every maximal order in  $B$  admits an embedding of  $\mathcal{O}_L$ . We suppose that  $\mathcal{R}$  is our fixed maximal order containing  $\mathcal{O}_L$ . That  $B_\nu$  is a division algebra means  $\mathcal{R}_\nu$  is the unique maximal order in  $B_\nu$ , whose normalizer is all of  $B_\nu^\times$  and so  $K_\nu^\times \subset H_{\mathcal{R}}$ . As we have seen this implies that  $\nu$  splits completely in the class field  $K(\mathcal{R})$ , and hence also in  $L_0 = K(\mathcal{R}) \cap L$ .

By the Albert-Brauer-Hasse-Noether theorem,  $m_\nu = n \mid [L_\mathfrak{P} : K_\nu]$  for all  $\mathfrak{P} \mid \nu\mathcal{O}_L$ . But this implies there is a unique prime  $\mathfrak{P}$  of  $L$  lying above  $\nu$ , and  $n = [L_\mathfrak{P} : K_\nu] = e(\mathfrak{P}|\nu)f(\mathfrak{P}|\nu)$ . Now  $\mathfrak{P}_0 = \mathfrak{P} \cap L_0$  is the unique prime of  $L_0$  lying above  $\nu$ , so  $[L_0 : K] = e(\mathfrak{P}_0|\nu)f(\mathfrak{P}_0|\nu)$ . But  $\nu$  splits completely in  $L_0$  which implies  $[L_0 : K] = 1$ . The result is now immediate from Corollary 4.2.  $\square$

What remains to resolve is the question of how partial ramification affects selectivity. We demonstrate by means of an explicit example that selectivity can occur in a partially ramified algebra.

*Example 5.3.* To compute our example, we employ Magma [6]. We construct a central simple algebra  $B$  of degree 16 over  $K = \mathbb{Q}(\sqrt{-14})$  which is ramified at two finite primes  $K$  so that at each of those primes  $\mathfrak{P}$ ,  $B_\mathfrak{P} \cong M_2(D_\mathfrak{P})$  with  $D_\mathfrak{P}$  a quaternion division algebra over  $K_\mathfrak{P}$ . We construct a field  $L$  which is contained in  $B$  and for which  $\mathcal{O}_L$  is selective. By Corollary 4.2, we need only show that  $L_0 = K(\mathcal{R}) \cap L \neq K$ . In the example we construct, the class field  $K(\mathcal{R})$  is simply the Hilbert class field of  $K$ .

The following calculations were produced by Magma [6]. Let  $K = \mathbb{Q}(\sqrt{-14})$ . The Hilbert class field of  $K$  is  $H_K = K(\alpha)$  where  $\alpha^4 + 4\alpha^2 - 28 = 0$ , and the ideal class group of  $K$ ,  $C_K$ , is isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ .

Let  $L_0 = K(\beta)$  where  $\beta^2 + (4\sqrt{-14} + 22)\beta + (44\sqrt{-14} + 33) = 0$ , and let  $L = L_0(\sqrt{-5})$ ; then  $L \cap H_K = L_0 \neq K$ . Let  $\mathfrak{P}_{137,1}, \mathfrak{P}_{137,2}$  be the two primes of  $K$  lying above 137 in  $\mathbb{Z}$ . One checks they are principal in  $K$ , so their classes  $[\mathfrak{P}_{137,i}]$  are trivial in  $C_K$ , and each prime splits completely in  $H_K$ .



Let  $B$  be the central simple algebra over  $K$  defined by the data:  $\dim_K B = 16$ ;  $Ram(B) = \{\mathfrak{P}_{137,1}, \mathfrak{P}_{137,2}\}$ ; and  $m_{\mathfrak{P}_{137,i}} = 2$  for  $i = 1, 2$ , that is  $B_{\mathfrak{P}_{137,i}} \cong M_2(D_{\mathfrak{P}_{137,i}})$  with  $D_{\mathfrak{P}_{137,i}}$  a quaternion division algebra over  $K_{\mathfrak{P}_{137,i}}$ , and is split at every other prime of  $K$ . The existence of  $B$  is guaranteed by short exact sequence of Brauer groups induced by the Hasse invariants (see e.g., (32.13) of [19]). To see that  $L$  embeds in  $B$ , we need only verify the conditions of the Albert-Brauer-Hasse-Noether theorem above. Since  $m_{\nu} = 1$  for all primes except  $\mathfrak{P}_{137,1}, \mathfrak{P}_{137,2}$ , we simply need to observe that  $m_{\mathfrak{P}_{137,i}} = 2 \mid [L_Q : K_{\mathfrak{P}_{137,i}}]$  for every prime  $Q$  of  $L$  lying above  $\mathfrak{P}_{137,i}$  which is verified by Magma which shows (by factoring) that each prime is unramified and has inertia degree 2 in  $L$ .

Then for a maximal order  $\mathcal{R} \subset B$ , with  $\mathcal{O}_L \subset \mathcal{R}$ , the associated class field has Galois group  $Gal(K(\mathcal{R})/K) \cong I_K/(P_K I_K^4) \cong C_K$ , since  $[\mathfrak{P}_{137,i}]$  is trivial. So  $G = Gal(K(\mathcal{R})/K) = Gal(H_K/K) \cong C_K$ . Since  $K(\mathcal{R}) \subseteq H_K$  we have equality, and selectivity is demonstrated.

We continue this example and give our parametrization as in Equations (6) - (8). Put  $H = Gal(K(\mathcal{R})/L_0) = \widehat{H}$ . Then  $Gal(L_0/K) \cong G/H = \langle \sigma H \rangle$  where  $\sigma$  is the Artin symbol  $(\mathfrak{P}_3, H_K/K)$  with  $\mathfrak{P}_3$  is the prime of  $K$  with  $\mathfrak{P}_3 \cap \mathbb{Z} = 3\mathbb{Z}$ . Finally choose  $\mathfrak{P}_7$  with  $7\mathcal{O}_K = \mathfrak{P}_7^2$ . Then

$$f(\mathfrak{P}_7; L_0/K) = 1; \quad f(\mathfrak{P}_7; K(\mathcal{R})/K) = 2; \quad f(\mathfrak{P}_7; L/K) = 1.$$

So  $H = Gal(K(\mathcal{R})/L_0) = \langle (\mathfrak{P}_7; K(\mathcal{R})/K) \rangle$ .

## REFERENCES

1. Luis Arenas-Carmona, *Applications of spinor class fields: embeddings of orders and quaternionic lattices*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 7, 2021–2038. MR MR2044166 (2005b:11044)
2. ———, *Representation fields for commutative suborders*, Ann. Inst. Fourier (Grenoble) (to appear) (arXiv:1104.1809v1).
3. Emil Artin and John Tate, *Class field theory*, AMS Chelsea Publishing, Providence, RI, 2009, Reprinted with corrections from the 1967 original. MR MR2467155 (2009k:11001)
4. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR 0242802 (39 #4129)
5. Cristina M. Ballantine, John A. Rhodes, and Thomas R. Shemanske, *Hecke operators for  $GL_n$  and buildings*, Acta Arithmetica **112** (2004), 131–140.
6. Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR MR1484478
7. Kenneth S. Brown, *Buildings*, Springer-Verlag, New York, 1989. MR MR969123 (90e:20001)
8. Wai Kiu Chan and Fei Xu, *On representations of spinor genera*, Compos. Math. **140** (2004), no. 2, 287–300. MR MR2027190 (2004j:11035)
9. C. Chevalley, *Algebraic number fields*, L'arithmétique dan les algèbres de matrices, Herman, Paris, 1936.
10. Ted Chinburg and Eduardo Friedman, *An embedding theorem for quaternion algebras*, J. London Math. Soc. (2) **60** (1999), no. 1, 33–44. MR MR1721813 (2000j:11173)
11. P. Doyle, B. Linowitz, and J. Voight, *Minimal isospectral and nonisometric 2-orbifolds*, (preprint).
12. Xuejun Guo and Hourong Qin, *An embedding theorem for Eichler orders*, J. Number Theory **107** (2004), no. 2, 207–214. MR MR2072384 (2005c:11141)

13. P. J. Higgins, *Introduction to topological groups*, Cambridge University Press, London, 1974, London Mathematical Society Lecture Note Series, No. 15. MR MR0360908 (50 #13355)
14. Serge Lang, *Algebraic number theory*, second ed., Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994. MR MR1282723 (95f:11085)
15. B. Linowitz, *Selectivity in quaternion algebras*, J. of Number Theory **132** (2012), 1425–1437.
16. B. Linowitz and T. Shemanske, *Embedding orders into central simple algebras*, to appear, *Journal de théorie des nombres de Bordeaux* (2012), <http://arxiv.org/pdf/1006.3683>.
17. C. Maclachlan, *Optimal embeddings in quaternion algebras*, J. Number Theory **128** (2008), 2852–2860.
18. Władysław Narkiewicz, *Elementary and analytic theory of algebraic numbers*, second ed., Springer-Verlag, Berlin, 1990. MR MR1055830 (91h:11107)
19. I. Reiner, *Maximal orders*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London-New York, 1975, London Mathematical Society Monographs, No. 5. MR MR0393100 (52 #13910)
20. Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237 (82e:12016)
21. Thomas R. Shemanske, *Split orders and convex polytopes in buildings*, J. Number Theory **130** (2010), no. 1, 101–115. MR MR2569844
22. Marie-France Vignéras, *Variétés riemanniennes isospectrales et non isométriques*, Ann. of Math. (2) **112** (1980), no. 1, 21–32. MR 584073 (82b:58102)

DEPARTMENT OF MATHEMATICS, 6188 KEMENY HALL, DARTMOUTH COLLEGE, HANOVER, NH 03755

*E-mail address*: `benjamin.linowitz@dartmouth.edu`

*URL*: <http://www.math.dartmouth.edu/~linowitz/>

DEPARTMENT OF MATHEMATICS, 6188 KEMENY HALL, DARTMOUTH COLLEGE, HANOVER, NH 03755

*E-mail address*: `thomas.r.shemanske@dartmouth.edu`

*URL*: <http://www.math.dartmouth.edu/~trs/>